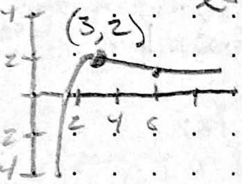
Ex 1: Derivative @ Rel. Extrema

ⓐ $f(x)=\frac{9(x^2-3)}{x^3}$ $f'(x)=\frac{x^3(9(x^2-3)(2x))-(9(x^2-3))(3x^2)}{x^6}$

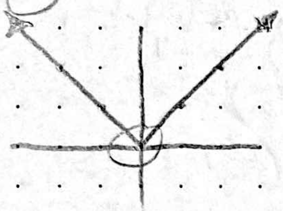


$f'(3)=0$

$(m @ x=3)=0$

maximum of 2 @ $x=3$.

ⓑ $f(x)=|x|$

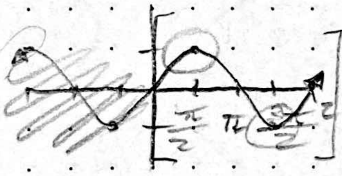


minimum of "0" @ $x=0$

no slope → cusp / sharp curve

not differentiable @ $x=0$

ⓒ $f(x)=\sin x$ on $[0, 2\pi]$ $f'(x)=\cos x$



max of 1 @ $x=\frac{\pi}{2}$ $f'(x)=0$ @ $x=\frac{\pi}{2}, \frac{3\pi}{2}$

min of -1 @ $x=\frac{3\pi}{2}$

3.3 Increasing & Decreasing Functions & 1st $\frac{dy}{dx}$ test (Oct. 24. 2024)

Definition of Increasing & Decreasing Functions:

A function is increasing on the interval x_1 & x_2 if $x_1 < x_2$ implies $f(x_1) < f(x_2)$

A function is decreasing on x_1 & x_2 if $x_1 < x_2$, implies $f(x_1) > f(x_2)$

Test for increasing / decreasing functions:

if on interval $[a, b]$ continuous & differentiable on (a, b):

1: if $f'(x)>0$ for all x on (a,b), f is increasing on $[a, b]$

2: if $f'(x)<0$ for all x on (a,b), f is decreasing on $[a, b]$

3: if $f'(x)=0$ for all x on (a,b), f is constant on $[a, b]$

Ex 1:

$f(x)=x^3-4x$ $f'(x)=3x^2-4$ $0=3x^2-4 \rightarrow 4=3x^2$

$\rightarrow \frac{4}{3}=x^2 \rightarrow x=\pm\sqrt{\frac{4}{3}} \rightarrow x=\pm\frac{2}{\sqrt{3}}$

$(-\infty, -\frac{2}{\sqrt{3}})\ (-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})\ (\frac{2}{\sqrt{3}}, \infty)$ | $f(x)$ is increasing on $(-\infty, -\frac{2}{\sqrt{3}}) \cup (-\frac{2}{\sqrt{3}}, \infty)$.

-3 0 3

ⓕ' + − + | $f(x)$ is decreasing on $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$

Ex 2: $f(x) = x^3 - \frac{3}{2}x^2$ $f'(x) = 3x^2 - 3x$

$0 = 3x^2 - 3x \rightarrow 0 = 3x(x-1) \rightarrow x = 0, 1$

$(-\infty, 0)(0,1)(1, \infty)$
$-1 \quad \frac{1}{2} \quad 2$
f' $+ \quad - \quad +$

$f(x)$ is increasing on $(-\infty, 0) \cup (1, \infty)$
$f(x)$ is decreasing on $(0,1)$

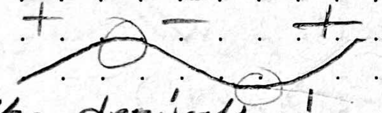
Monotonic functions - No mins/maxs., only increasing/decreasing

Ex: $(\cdot, \cdot)(\cdot, \cdot)$ with $+ \quad +$ or $(\cdot, \cdot)(\cdot, \cdot)$ with $- \quad -$

Ex 3: $f(x) = x^3 - 12x - 5$ $f'(x) = 3x^2 - 12$

$0 = 3x^2 - 12 \rightarrow 12 = 3x^2 \rightarrow 4 = x^2 \rightarrow x = \pm 2$

$(-\infty, -2)(-2, 2)(2, \infty)$
$-3 \quad 0 \quad 3$
f' $+ \quad - \quad +$



← Use derivative!

↑

Local (Relative) max is $\underline{11}$ @ $x = -2$.
Local (Relative) min. is $\underline{-21}$ @ $x = 2$.

$-2^3 - 12(-2) - 5 \quad -8 + 24 - 5 \rightarrow 11$

$2^3 - 12(2) - 5 \rightarrow 8 - 24 - 5 \rightarrow -21$

first derivative test!

if slope changes from + to −, or vice versa, there is a rel max or min.

3.2 Rolle's Theorem & Mean Value Theorem

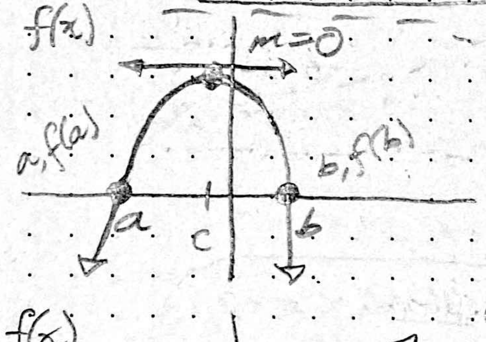
Oct. 25, 2024

Rolle's Theorem

f is continuous on [a, b] & differentiable on (a, b)

if $f(a) = f(b)$, ~~then~~ there is ~~at least 1~~ number "c" in (a, b) such that $f'(c) = 0$

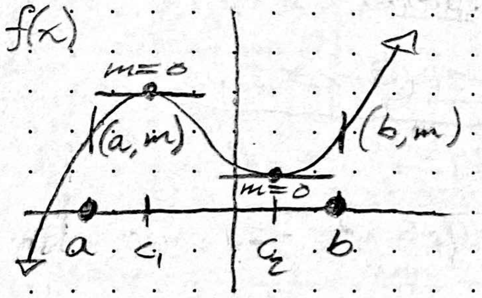
Ex



3 conditions:

- f(x) must be continuous on [a, b]
- f(x) must be differentiable on (a, b)
- $f(a) = f(b)$ (y-vals are equal)

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{n} = 0 \rightarrow$ (for Rolle's Theorem)



Ex 1: $f(x) = x^2 - 3x + 2$ ✓✓✓

$0 = x^2 - 3x + 2 \rightarrow 0 = (x-2)(x-1)$

$x = 1; 2$ $f'(x) = 2x - 3$

$0 = 2x - 3 \rightarrow 3 = 2x \rightarrow \boxed{x = 3/2 = c}$

Ex 2:

$f(x) = x^4 - 2x^2$ $f'(x) = 4x^3 - 4x$ on (-2, 2)

$f(-2) = (-2)^4 - 2(-2)^2 \rightarrow 16 - 8 \rightarrow 8$ $f(2) = (2)^4 - 2(2)^2 \rightarrow 16 - 8 \rightarrow 8$

differentiable ✓ continuous ✓ $f(a) = f(b)$ ✓

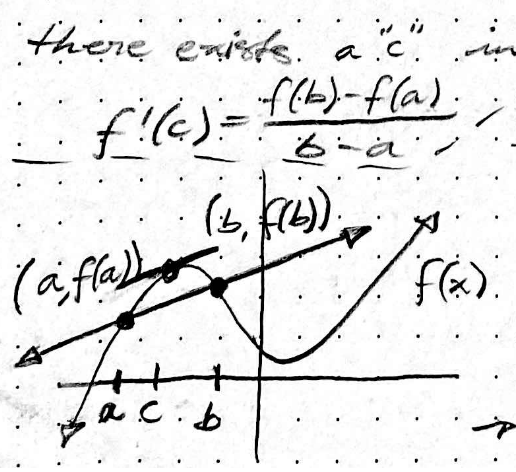
$0 = 4x^3 - 4x \rightarrow 0 = 4x(x^2 - 1) \rightarrow 0 = x \rightarrow 0 = x^2 - 1 \rightarrow x = \pm 1$

$\boxed{c = 0, \pm 1}$

Mean Value Theorem:

If f is continuous on [a, b] & differentiable on (a, b), then there exists a "c" in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Ex 3: $f(x) = 5 - \frac{4}{x}$ on $(\overset{a}{1}, \overset{b}{4})$

$f(1) = 5 - 4 \rightarrow 1 \rightarrow (1, 1)$ $f(4) = 5 - 1 \rightarrow 4 \rightarrow (4, 4)$

- continuous ✓

$f'(x) = \frac{-x(0) + 4(1)}{x^2} \rightarrow \boxed{\frac{4}{x^2}}$ $\frac{f(4) - f(1)}{4 - 1} = \frac{4}{x^2}$

$\rightarrow \frac{4-1}{4-1} = \frac{4}{x^2} \rightarrow 1 = \frac{4}{x^2} \rightarrow x = \pm 2$ ★ $\boxed{c = 2}$ ★

③.4 Concavity & 2nd derivative test

Oct. 29. 2024

WU: $f(x)=\frac{x-1}{x^2} \rightarrow f(x)=\frac{x^2(1)-(x-1)(2x)}{x^4} \rightarrow \frac{x^2-2x^2+2x}{x^4} \rightarrow \frac{-x^2+2x}{x^4}$

$0=\frac{-x^2+2x}{x^4} \rightarrow \boxed{0=-x^2+2x} \rightarrow 0=x(-x+2) \rightarrow \boxed{\begin{matrix} x=0=c \\ x=2 \end{matrix}}$

$0=-x+2 \rightarrow$

Asymptote

$(-\infty,\boxed{0})(0,2)(2,\infty)$

$-1 \quad 1 \quad 3$

$\boxed{f'(x)}$ $- \quad + \quad -$

$\frac{-(-1)^2+2(-1)}{(-1)^4} \rightarrow -1-2 \rightarrow -3$

$\frac{-(1)^2+2(1)}{(1)^4} \rightarrow \frac{-1+2}{1} \rightarrow 1$

$\frac{-(3)^2+2(3)}{3^4}$

$\frac{-9+6}{}$

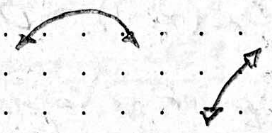
f(x) increases on (0,2)   f(x) decreases on $(-\infty,0) \cup (2,\infty)$

~~$f(0)=\frac{0-1}{0^2}$ undefined~~ | $f(2)=\frac{2-1}{4} \rightarrow \boxed{\frac{1}{4}} \rightarrow \boxed{(2,\frac{1}{4}) \text{ relative maximum}}$

Concavity → Hollowed / Curved Inward

concave up: "cup"

concave down: "frown"



Definition: Let f be differentiable on I. Graph of f is concave up on I when f' is increasing & f is concave down when f' is decreasing.

Theorem 3.7 Test for concavity

If $f''(x)>0$ for all I, then $f(x)$ is concave upward.

If $f''(x)<0$ for all x in I, then $f(x)$ is concave downward.

Ex 1: ⓐ $y=3+\sin(x)$ on $(0,2\pi)$

$y'=\cos(x) \quad y''=-\sin(x) \quad 0=-\sin(x) \quad x=\pi$

$(0,\pi)(\pi,2\pi)$

$\pi/2 \quad 3\pi/2$

$\boxed{y''}$ $- \quad +$

concave down   concave up

f(x) is concave up on $(\pi,2\pi)$.

f(x) is concave down on $(0,\pi)$.

ⓑ $y=4x^3+21x^2+36x-20 \rightarrow y'=12x^2+42x+36 \quad \boxed{y''=24x+42}$

$0=24x+42 \rightarrow -42=24x \rightarrow x=\frac{-42}{24} \rightarrow x=-\frac{7}{4}$

$(-\infty,-\frac{7}{4})(-\frac{7}{4},\infty)$

$-2 \quad 0$

$\boxed{f''}$ $- \quad +$

concave down   concave up

$24(-2)+42 \rightarrow -48+42 \rightarrow -6$

$24(0)+42 \rightarrow 42$

f(x) is concave up on $(-\frac{7}{4},\infty)$ & concave down on $(-\infty,-\frac{7}{4})$.

Point of

Point of inflection: Vertical tangent line of function where concavity switches

→ Point! Full coordinate required! → (x, y)

Theorem 3.8 Points of Inflection

If $(c, f(c))$ is point of inflection on the graph of f, then either $f''(c) = 0$ or $f''(c)$ doesn't exist @ $x = c$

Ex 2: $f(x) = \cos x$ on $[0, 2\pi]$ find points of inflection

$f'(x) = -\sin x$ $\boxed{f''(x) = -\cos x}$ $0 = -\cos x \to \pi/2, 3\pi/2$

Point #1: $(\pi/2, 0)$ Point #2: $(3\pi/2, 0)$

The points of inflection are @ $(\frac{\pi}{2}, 0)$ & $(\frac{3\pi}{2}, 0)$.

Ex 3: $f(x) = 3x^5 - 5x^4 + 1 \to f'(x) = 15x^4 - 20x^3$ $f''(x) = 60x^3 - 60x^2$

$0 = 60x^3 - 60x^2 \to 0 = 60x^2(x-1) \to 0 = x^2 \to 0 = x$ $0 = x - 1$ $1 = x$

$c = 0, 1$

$(-\infty, 0)(0,1)(1, \infty)$ point of inflection is $(1, -1)$.

-1 $1/2$ 2

f'' $-$ $-$ $+$ $f(x)$ is concave down on $(-\infty, 1)$ & concave up on $(1, \infty)$.

Second Derivative Test (Theorem 3.9)

f is a function such that $f'(c) = 0$ & $f''(x)$ exists on "I" including "c" $\leftarrow (x_1, x_2)$

- if $f''(c) > 0$ then f has rel min @ $(c, f(c))$
- if $f''(c) < 0$ then f has rel max @ $(c, f(c))$

If $f''(c) = 0$, the test fails; use the first derivative test.

Ex 4: $f(x) = x^3 - 12x - 5 \to f'(x) = 3x^2 - 12 \to 0 = 3x^2 - 12$

$12 = 3x^2 \to 4 = x^2$

$f''(x) = 6x \to f''(2) = 6(2) \to +12 (>0)$ concave up → rel. min $x = \pm 2$

$f''(-2) = 6(-2) \to -12$ concave down → rel. max.

There is a relative minimum of -21 @ $x = 2$.
There is a relative maximum of " @ $x - 2$.

HW:

3.5 Limits at Infinity (End Behavior)

Oct. 31. 2024

Warm Up $f(x)=x^4-x^3-3x^2-2$

$f'(x)=4x^3-3x^2-6x \rightarrow f''(x)=12x^2-6x-6 \rightarrow 0=12x^2-6x-6 \rightarrow x^2-6x-72$

$0=(x-12)(x+6) \rightarrow 0=(x-1)(x+\frac{1}{2})$ $x=-\frac{1}{2},1$

$(-\infty,-\frac{1}{2})(-\frac{1}{2},1)(1,\infty)$ $f(-\frac{1}{2})=$

f'' $\overset{-1}{+}$ $\overset{0}{-}$ $\overset{3}{+}$

Inflections: $(-\frac{1}{2},-\frac{41}{16})$ & $(1,-5)$

Concave up on $(-\infty,-\frac{1}{2})\cup(1,\infty)$. Concave down on $(-\frac{1}{2},1)$.

What is a limit @ infinity? End behavior of a function's graph (in limit notation).

Ex: left side $\lim_{x\to-\infty} f(x)=$ right side $\lim_{x\to\infty} f(x)=$

How do you find a horizontal asymptote?

for $f(x)=\frac{ax^n}{bx^m}$

- if $n<m$, horizontal asymptote is $y=0$
- if $n=m$, horizontal asymptote is $y=\frac{a}{b}$
- if $n>m$, there is no horizontal asymptote
- if $n>m$ by exactly 1, there is a slant asymptote

Definition of a Horizontal Asymtote:

The line $y=L$ is a horizontal asymptote on "f" if

$\lim_{x\to\infty} f(x)=L$ and/or $\lim_{x\to-\infty} f(x)=L$

Ex 1:

① H.A. is $y=0$ $\lim_{x\to\infty} f(x)=0$

② H.A. is $y=\frac{2}{5}$ $\lim_{x\to-\infty} f(x)=\frac{2}{5}$

③ no H.A. $\lim_{x\to\infty} f(x)=\infty$ $\frac{2x^5}{3x^2}\rightarrow\frac{2}{3}x^3$

④ no H.A $\lim_{x\to-\infty} f(x)=-\infty$

for 3&4, divide leading terms to find term for polynomial rules of End Behavior

Theorem 3.10 Limits @ infinity

if "r" is positive & "c" is real, then $\lim_{x \to \infty} \frac{c}{x^r} = 0.$

And if x^r is defined when $x < 0$, then $\lim_{x \to -\infty} \frac{c}{x^r} = 0.$

Ex 2: $\lim_{x \to \infty} (5) - \lim_{x \to \infty} \left(\frac{2}{x^2}\right) \to 5 - 0 = \boxed{5}$

or translate: $\lim_{x \to \infty} \left(\frac{5x^2 - 2}{1x^2}\right) = 5$

Ex 3:

$\lim_{x \to \infty} \frac{2x-1}{1x+1} = 2$

Ex 4:

even-even-odd $\begin{cases} \sqrt[4]{x^{12}} = x^{3} \\ \sqrt[4]{x^{12}} = \pm x \text{ or } |x| \end{cases}$

ⓐ $\lim_{x \to \infty} \frac{3x-2}{\sqrt{2x^2+1}}$ HA $= \frac{3}{\sqrt{2}}$ $y = \frac{3}{\sqrt{2}}$

if $\lim_{\to \infty}$, use +

ⓑ $\lim_{x \to -\infty} \frac{3x-2}{\sqrt{2x^2+1}}$ HA $= \frac{3}{-\sqrt{2}}$ $y = \frac{3}{-\sqrt{2}}$

if $\lim_{x \to -\infty}$, use −

$\sqrt{x^2} = \pm x$

Ex 5:

ⓐ $\lim_{x \to \infty} \sin x =$ DNE, due to oscillation

ⓑ $\lim_{x \to \infty} \frac{\sin x}{x} = 0$ squeeze theorem

202 HW: Q: 1-6, 13, 17, 21, 23, 27, 31, 33, 35

3.6 Curve Sketching

Sep. 5. 2024

WU: ① $\lim_{x\to-\infty} f(x)=+\infty$ ② $\lim_{x\to-\infty} f(x)=2$

③ $\lim_{x\to\infty} f(x)=-\infty$ ④ $\lim_{x\to\infty} f(x)=\frac{3}{2}$ ⑤ $\lim_{x\to\infty} f(x)=\frac{\sqrt{3}}{2}$

Guidelines for Analyzing a Function & How to Sketch it:

HW: 9, 14, ㉑ Start

• x-int. & y-int.:

x: set function equal to 0 & solve

y: set all x's to 0s in function, simplify

• domain & range:

D: Set of all x-vals in f(x)

R: Set of all y-vals in f(x)

• Vertical asymptotes

check for holes in function, if cannot be removed = VA

or check what makes denom. 0

• relative extrema (local)

minimum or maximum of f(x) (find y-val)

• Hole

point that is undefined on f(x)

common factor between numerator & denominator

• point of inflection

point where concavity switches from up to down or vice versa

• symmetry:

when a graph is mirrored over an axis:

→ flipped on y-axis

→ flipped on Origin

• continuity:

every x is defined, no removable or non-removable discontinuities

• differentiability

try to take derivative of function

no cusps

• Horizontal Asymptote

$n<m$ $y=0$

$n=m$ $y=a/b$

$n>m$ $y=$ no HA

line f(x) approaches a $x\to\pm\infty$

• Concavity

if $f''(x)>0$: concave up

if " <0: concave down

• Limits @ infinity

limit as x approaches $\pm\infty$

$\lim_{x\to\pm\infty} f(x)=$ H.A. or $\lim_{x\to\pm\infty} f(x)=$ polynomial end behavior rules

Ex 1: $f(x) = x^3 - 4x$ $= x(x^2-4)$ $\Rightarrow x = 0, \pm 2$

$f'(x) = 3x^2 - 4$ $f''(x) = 6x = 0$ x-int: $0, \pm 2$ y-int: 0

VA: no VA HA: no HA Critical numbers: $\pm\frac{2}{\sqrt{3}}$ (from $f'(-x)$)

$(-\infty, -\frac{2}{\sqrt{3}})$ $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ $(\frac{2}{\sqrt{3}}, \infty)$ relative minimum is -3.079 @ $x = \frac{2}{\sqrt{3}}$

$\boxed{f'}$ -2 $+$ 0 $-$ 2 $+$ relative maximum is 3.079 @ $x = -\frac{2}{\sqrt{3}}$

rel max rel min

$(-\infty, 0)$ $(0, \infty)$ $f(x)$ is concave down on $(-\infty, 0)$.

f'' -1 $-$ inflec at 0 1 $+$ $f(x)$ is concave up on $(0, \infty)$.

con

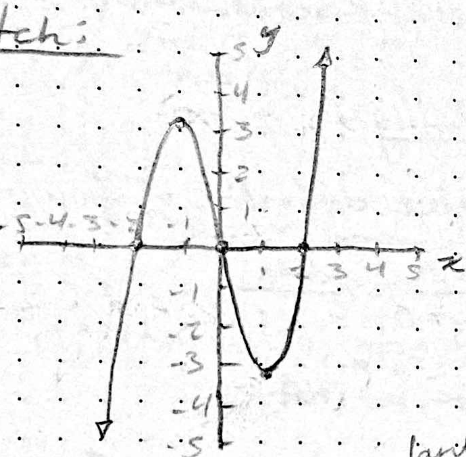
point of inflection is @ $(0,0)$

domain: $(-\infty, \infty)$, All real numbers Symmetry: $f(-x) = -x^3 - 4(-x)$

$= -x^3 + 4x$

$f(-x) = -f(x) \Rightarrow$ odd!

Sketch:



if $b^2 - 4ac > 0$, there are real zeros

slant @

Ex 2: $f(x) = \frac{x^2 - 2x + 4}{x - 2}$ $f'(x)$: $\frac{(x-1)(2x-2) - (x^2-2x+1)(1)}{(x-2)^2}$

$\underline{2}|$ 1 -2 4

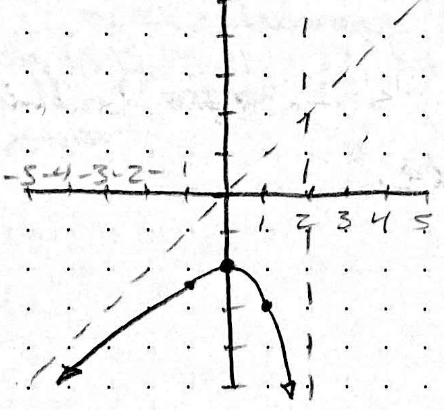
$\downarrow$ 2 0

1 0 $\cancel{4}$

$x - 2 = 0 \Rightarrow x = 2$

x-int: $(-2)^2 - 4(1)(9) < 0$ no real zeros

y-int: $\frac{0 - 0 + 4}{0 - 2}$ $-2 \Rightarrow (0, -2)$ VA: $x = 2$



x	y
1	-3
-1	-2.33
3	7
4	6
3.5	6.166

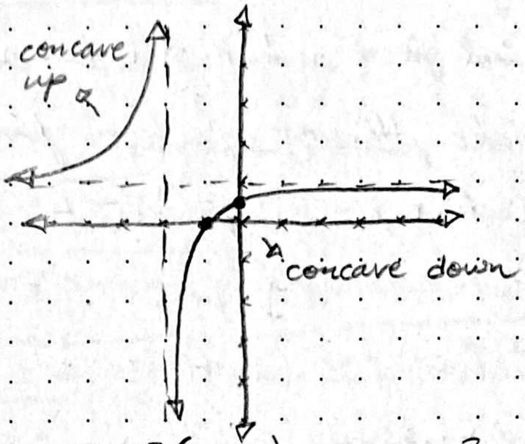
3.7 Optimization

Nov 12 2024

Warm-up $y=\frac{x+1}{x+2}$ $0=x+1$

$x=-1$

$y=\frac{1}{2}$



$y'=\frac{x+2(1)-(x+1)(1)}{(x+2)^2}$

$y'=\frac{x+2-x-1}{(x+2)^2} \to y'=\frac{1}{(x+2)^2}$

$y''=\frac{(x+2)^2(0)-(1)2(x+2)(1)}{(x+2)^4} \to \frac{-2x-4}{(x+2)^4} \to \frac{-2(x+2)}{(x+2)^2(x+2)^2} \to \frac{-2}{(x+2)^3}$

$0=\frac{1}{(x+2)^2}$ no critical numbers

$x \neq -2$ $(-\infty,-2)(-2,\infty)$ no relative extrema

$\textcircled{f'}$ $\overset{-3}{+}$ $\overset{0}{+}$

$0=\frac{-2}{(x+2)^3} \to$ no critical values $(-\infty,-2)(-2,\infty)$

$x \neq -2$

$\textcircled{f''}$ $\overset{-3}{+}$ $\overset{0}{-}$

up down

*Check whether tests ask for extrema x-val. or actual point.

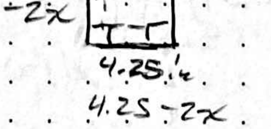
Optimization - The process of maximizing or minimizing.

- for calculus: finding the mins. & maxs.

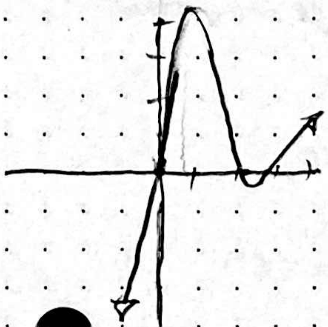
Ex1: Making a box, no top 5.5 in × 4.25 in sheet cardboard

how big can the ~~box~~ be made if cutting squares of x size out of the corners?

$V=L \times W \times H$



$V=(5.5-2x)\times(4.25-2x)\times(x)$ $h=x$



$max=0.793$ in

The squares should be 0.793 in. long to hold a maximum volume of 8.269 in^3.

Solving Minimum & Maximum Problems

1. Find given quantities to be determined. Make a sketch.
2. Make primary equation for quantity to be maximized or minimized
3. Reduce prim. equation to one with 1 independent variable.
   → Maybe secondary equations
4. Determine feasible domain of prime equation. Determine values where problem makes sense
5. Determine max or min values through calc.

Ex 2: 2 positive nums, x & y

The product is 192 & the sum of x & $3y$ is a min

$x \times y = 192$ $\qquad$ $\underbrace{x + 3y = \text{min}}_{\text{minimize}}$ on graph or "S"

Don't use $+\infty$ for this problem → feasible domain

$S(x) = x + 3y \rightarrow 192/x = y$

$S(x) = x + 3\left(\frac{192}{x}\right) \rightarrow S(x) = x + 576x^{-1}$

$S'(x) = 1 - 1(576)x^{-2} \rightarrow S'(x) = 1 - \frac{576}{x^2}$

$0 = 1 - \frac{576}{x^2} \rightarrow 1 = \frac{576}{x^2} \rightarrow x^2 = 576 \rightarrow x = \pm 24$

use only +24 $\quad x = 24$

	(0,24)	(24, 192)
	1	25
s'(x)	−	+

rel min.

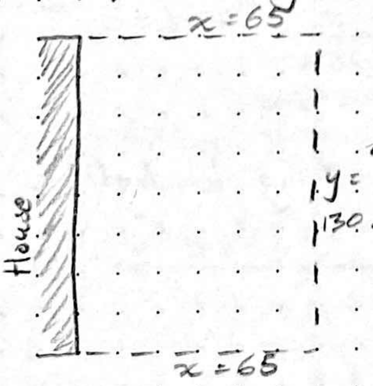
find y → $24 \times y = 192$ → $y = 8$

$y = 8$

The 2 positive numbers that minimize the sum are 24 & 8.

Ex 3: 260 ft material. Rect shape. Only 3 sides of rect

$A = x \times y$ $\qquad$ $260 = 2x + y$ → maximize the area



$A(x) = x(260 - 2x)$ ← $\boxed{260 - 2x = y}$

$A(x) = 260x - 2x^2$

	(0, 65)	(65, 260)
	1	66
$\boxed{f'}$	+	−

max @ $x = 65$

$A'(x) = 260 - 4x$

$0 = 260 - 4x$ $\qquad$ $260 = 130 + y$

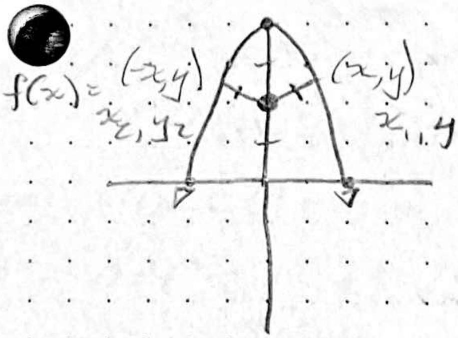
$4x = 260$ $\qquad$ $y = 130$

$x = 65$

The dimensions of the fence that maximizes its area are 65 ft x 130 ft.

Ex 4: Which points on $y = 4-x^2$ are closest to $(0,2)$?

minimize → closest $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$



minimize distance

$d = \sqrt{(x-0)^2+(y-2)^2}$

$d = \sqrt{(x-0)^2+((4-x^2)-2)^2}$

$d = \sqrt{x^2+(4-x^2-2)^2} \rightarrow \sqrt{x^2+(2-x^2)^2}$

$d = \sqrt{x^4-3x^2+4}$

$\frac{d}{dx}\left(x^4-3x^2+4\right) = 4x^3-6x = 0$

$2x(2x^2-3) = 0$

$x=0$ $x = \pm\sqrt{\frac{3}{2}}$

$(-\infty, -\sqrt{\frac{3}{2}})(-\sqrt{\frac{3}{2}}, 0)(0, \sqrt{\frac{3}{2}})(\sqrt{\frac{3}{2}}, \infty)$

$-$ $+$ $-$ $+$

min @ $x = \pm\sqrt{\frac{3}{2}}$

The closest points to $(0,2)$ are $\left(\pm\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$.

HW: 4, 6, 9, 11, 16, 19, 25

Chapter 3 Review

① $f(x) = x^2+5x,\ [-4,0]$ $f'(x) = 2x+5$ $0 = 2x+5 \to -5 = 2x \to x = -\frac{5}{2}$

$-4^2+5(-4) \to \overset{x=-4}{16-20 \to \boxed{-4}}$ $\overset{x=0}{0}$ $\left(-\frac{5}{2}\right)^2+5\left(-\frac{5}{2}\right)^{x=-\frac{5}{2} \text{ or } -\frac{10}{4}} = -\frac{25}{4}$

Absolute maximum of 0 @ $x=0$.

Absolute minimum of $-\frac{25}{4}$ @ $x = -\frac{10}{4}$.

⑦ $g(x) = 2x + 5\cos x,\ [0, 2\pi]$ $g'(x) = 2 - 5\sin x$

$0 = 2 - 5\sin x \to 5\sin x = 2 \to \sin x = \frac{2}{5}$ $y = 0.88$ $x = 2.73$

$0+5(1) \overset{x=0}{=} 5$ $4\pi + 5\cos(2\pi) \to 4\pi + 5 \to 4\pi + 5 \overset{x=2\pi}{\to} 17.566$

Abs minimum of 0.88 @ $x = 2.73$

Abs max of 17.566 @ $x = 2\pi$

⑨ $f(x) = 2x^2-7,\ [0,4]$ $f'(x) = 4x$ $f(0) = 0-7 \to -7$
$f(4) = 8 - 7 \to 7$

$f(0) \neq f(4)$, Rolle's Theorem cannot be applied.

⑪ $f(x) = \frac{x^2}{1-x^2},\ [-2,2]$ $f(-2) = \frac{(-2)^2}{(1-(-2)^2)} \to \frac{4}{-3} \to -\frac{4}{3}$

$f(-2) = f(2)$ $f(2) = \frac{(2)^2}{1-(2)^2} \to \frac{4}{-3} \to -\frac{4}{3}$

$f'(x) = \frac{(1-x^2)(2x)-(x^2)(-2x)}{(1-x^2)^2} \to \frac{2x - 2x^3 + 2x^3}{(1-x^2)^2} \to \frac{2x}{(1-x^2)^2}$

$0 = \frac{2x}{(1-x^2)^2} \to 0 = 2x \to \boxed{x=0}$ $f(0) = \frac{0}{1-0} \to \boxed{0}$ $\boxed{c = (0,0)}$

not continuous on $[-2,2]$, Rolle's theorem does not apply.

⑬ $f(x) = x^{2/3},\ [\overset{a}{1},\overset{b}{8}]$ continuous ✓ $f'(x) = \frac{2}{3}x^{-1/3}$ diff-able ✓

$\frac{2}{3}x^{-1/3} = \frac{f(8)-f(1)}{8-1}$ $f(1) = 1$ $f(8) = 4$

$\frac{2}{3}x^{-1/3} = \frac{4-1}{8-1} \to \frac{2}{3}x^{-1/3} = \frac{3}{7} \to x^{-1/3} = \frac{3}{7} \cdot \frac{3}{2} \to x^{-1/3} = \frac{9}{14}$

$\frac{1}{\sqrt[3]{x}} \to 1 = \frac{9}{14}\sqrt[3]{x} \to \frac{14}{9} = \sqrt[3]{x} \to x = 3.764$

㉑ $f(x) = x^2+3x-12 \to f'(x) = 2x+3$ $0 = 2x+3 \to x = \frac{-3}{2}$

$(-\infty, -\frac{3}{2})(-\frac{3}{2}, \infty)$ $f(x)$ is increasing on $(-\frac{3}{2}, \infty)$ and decreasing on $(-\infty, -\frac{3}{2})$.

	-2	0
f'	$-$	$+$

㉗ $f(x)=x^2-6x+5 \rightarrow f'(x)=2x-6 \quad 0=2x-6 \rightarrow x=\frac{6}{2} \rightarrow \boxed{x=3}$

$(-\infty, 3)(3,\infty)$ f(x) is inc. on $(3,\infty)$ and dec. on $(-\infty,3)$.

0 4

f' − + there is a relative minimum of −4 @ x=3.

$(3)^2-6(3)+5 \rightarrow 9-18+5 \rightarrow 14-18 \rightarrow -4$

㉙ $h(t)=\frac{1}{4}t^4-8t \rightarrow h'(t)=1t^3-8$

$0=t^3-8 \rightarrow t^3=8 \rightarrow t=2 \quad (-\infty,2)(2,\infty)$

$h(2)=\frac{1}{4}(2)^4-8(2)$

$\rightarrow 4-16$

0 3

h(x) is increasing on (2,∞) & f' − +

decreasing on (−∞,2). There is a relative minimum of −12 @ x=2.

㉛ $f(x)=\frac{x+4}{x^2} \rightarrow f'(x)=\frac{(x^2)(1)-(x+4)(2x)}{x^4} \rightarrow \frac{x^2-2x^2+8x}{x^4}$

$f'(x)=\frac{-x^2+8x}{x^4} \rightarrow 0=\frac{-x^2+8x}{x^4} \rightarrow -x^2+8x=0 \rightarrow x(-x+8)=0$

$\rightarrow x \neq 0$

~~$x=0$~~

$-x+8=0 \rightarrow 8=x$

$+x \quad +x$

㉝ $f(x)=\cos x-\sin x, \ (0,2\pi)$

$f'(x)=-\sin x-\cos x \rightarrow 0=-\sin x-\cos x \rightarrow 0=\sin x+\cos x$

$x=\frac{3\pi}{4}, \frac{7\pi}{4}$

sin = y-val

on unit

circle!

4.1 Antiderivatives & Indefinate Integration

3 Dec 2024

WU ① $f(x) = x^3 + 2x^2 - 5 \rightarrow f'(x) = 3x^2 + 4x$

② $f(x) = (2x+5)^5 \rightarrow f'(x) = 5(2x+5)^4(2) \rightarrow f'(x) = 10(2x+5)^4$

Antiderivative - A function that reverses what a derivative does.

↳ "Integral" ↓ sometimes very hard to find

Definite Integral - Area under the function's curve

Antiderivative

A function F is an antiderivative of f on an interval I when $F'(x) = f(x)$ for all x in I.

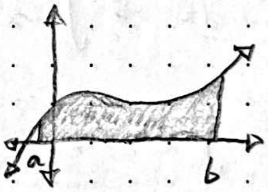
Indefinite Integral → no limits/boundaries, no a or b

$$\int f(x)\,dx = F(x) + C$$

↳ Ans is a function

Definite Integral → has a and b

$$\int_a^b f(x)\,dx = \text{Area / real number}$$



Ex:

$f(x) = x^3 + 4x + 1 \rightarrow f'(x) = 3x^2 + 4$

$f(x) = x^3 + 4x - 3$

$f(x) = x^3 + 4x + \frac{1}{2}$

All have the same derivative

+ C → Know 1 more point to find original

$$y = \int \underset{\text{Integrand}}{f(x)}\, \overset{\text{Variable of integration}}{dx} = \underset{\text{An antiderivative of } f(x)}{F(x)} + \overset{\text{Constant of integration}}{C}$$

Ex 1: $y' = 2 \rightarrow y = 2x + C$

$dx \cdot \frac{dy}{dx} = 2 \cdot dx$

$\rightarrow \int^{(1)} dy = \int 2\,dx$

$\rightarrow \boxed{y = 2x + C}$

Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$ Differentiation is the "inverse" of integration

Ex 2: $\int 3x^2\,dx \to 3\int x^2\,dx \to \frac{\cancel{3}(x^3)}{\cancel{3}} + C$

$\to x^3 + C = \int 3x^2\,dx$

Ex 3: find indefinite integral

ⓐ $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C$ or $\frac{1}{-2x^2} + C$

ⓑ $\int \sqrt{x^1}\,dx = \int x^{3/2}\,dx = \frac{2x^{3/2}}{3} + C$

ⓒ $\int 2\sin x\,dx = 2\int \sin x\,dx = -2\cos x + C$

Always +C for indefinite integrals!

Ex 4:

ⓐ $\int dx \to \int 1\,dx \to \int 1x^0\,dx = x + C$

ⓑ $\int (x+2)\,dx = \int x\,dx + \int 2\,dx = \frac{x^2}{2} + 2x + C$

Ex 5:

$\int \frac{x+1}{\sqrt{x}}\,dx \nearrow \frac{x+1}{\sqrt{x}} \to \frac{x+1}{x^{1/2}} \to (x+1)x^{-1/2} \to x^{1/2} + x^{-1/2}$

$= \int (x^{1/2} + x^{-1/2})\,dx = \frac{2x^{3/2}}{3} + 2x^{1/2} + C$

Ex 6: $\int \frac{\sin x}{\cos^2 x}\,dx \to \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} \to \tan x \cdot \sec x$

$\to \int \sec x \tan x\,dx = \sec x + C$

Ex 7:

ⓐ $\int \frac{2}{\sqrt{x^1}}\,dx \to 2x^{-1/2} \to \int 2x^{-1/2}\,dx = 4x^{1/2} + C$

ⓑ $\int (t^2+1)^2\,dt \to t^4 + 2t^2 + 1 \to \int (t^4 + 2t^2 + 1)\,dt = \frac{t^5}{5} + \frac{2}{3}t^3 \boxed{+t} + C$

ⓒ $\int \frac{x^3+3}{x^2}\,dx \to x + 3x^{-2} \to \int (x + 3x^{-2})\,dx = \frac{x^2}{2} + \frac{3x^{-1}}{-1} \to \frac{x^2}{2} - \frac{3}{x} + C$

ⓓ $\int \sqrt[3]{x}\,(x-4)\,dx = \int (x^{4/3} - 4x^{1/3})\,dx = \frac{3x^{7/3}}{7} - 4x^{4/3}\left(\frac{3}{4}\right) + C$

$= \frac{3x^{7/3}}{7} - 3x^{4/3} + C$

To find a particular solution,
you need an initial condition given as a point.

This means find C using the initial condition.

Ex 8: $F'(x) = \frac{1}{x^2}$, $x > 0$ find general solution.

$\frac{dy}{dx} = x^{-2} \rightarrow \int dy = \int x^{-2}\,dx \rightarrow y = \frac{x^{-1}}{-1} + C \rightarrow \boxed{y = -\frac{1}{x} + C} = F(x)$

find the particular solution that satisfies $F(1) = 0 \rightarrow (1, 0)$

$0 = \frac{-1}{1} + C \rightarrow 1 = C \rightarrow$ specific: $\boxed{y = -\frac{1}{x} + 1}$

Ex 9: Ball thrown up, initial velocity $\underset{V_0}{64}$ ft/s, initial height $\underset{S_0}{80}$ ft.

ⓐ find position function giving the height "s" as a function of time:

$s(t) = -16t^2 + 64t + 80$

$(s(t) = -16t^2 + V_0 t + S_0)$ = position func.

ⓑ When does the ball hit the ground?

$0 = -16t^2 + 64t + 80$

$0 = -16(t^2 - 4t - 5) \rightarrow 0 = t^2 - 4t - 5 \rightarrow (x - 5)(x + 1)$

negative time isn't an answer, eliminate

$\rightarrow x = 5, \cancel{-1} \rightarrow$ Ball hits the ground @ $t = 5$

HW: p 251; Qs: 5, 9, 13, 17, 21, 27, 33, 37, 50, 56.

4.4. A Fundamental Theorem of Calculus Dec. 5. 2024

WU ② $\int -\frac{7}{\cos^2 x}\,dx \rightarrow \int -7\sec^2 x\,dx \rightarrow -7\tan x + C$

⑨ $\int \frac{3\cos x}{\sin^2 x}\,dx \rightarrow 3\int \left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin x}\right)dx \rightarrow 3\int \cot x\csc x\,dx \rightarrow -3\csc x + C$

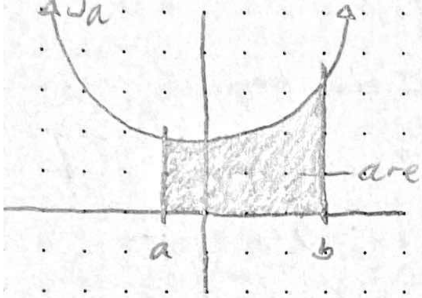
FTC (Fundamental...)

If function on $[a,b]$, & F is an antiderivative of f on $[a,b]$ the

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad = \text{real \# (area)}$$

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a) = \text{real number}$$

no + c



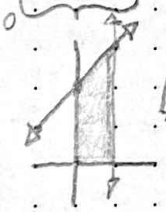
$\int_a^b f(x)\,dx$ = area of region enclosed by

- $f(x)$
- x-axis
- $x = a$
- $x = b$

Ex 1:

Trapazoid area:
$\frac{1}{2}h(b_1+b_2)$

$$\int_0^1 \underbrace{(x+2)}\,dx \rightarrow \overbrace{\frac{1}{2}(1)(2+3)}^{\text{Trap. area (Geometry)}} \rightarrow \frac{1}{2}(5) \rightarrow \boxed{\frac{5}{2}}$$

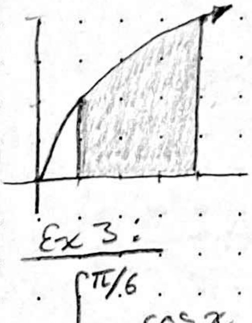


$$\int_0^1 (x+2)\,dx = \left[\frac{x^2}{2} + 2x\right]_0^1 = \left[\overset{f(b)}{\left(\frac{1}{2}+2\right)} - \overset{f(a)}{(0)}\right] = \boxed{\frac{5}{2}}$$

Calc way

Ex 2:

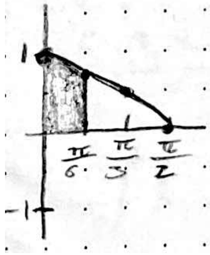
$$\int_1^4 \overset{2x^{1/2}}{2\sqrt{x}}\,dx \rightarrow \left[2\left(\frac{x^{3/2}}{3/2}\right)\right] \rightarrow \left[2\left(\frac{2x^{3/2}}{3}\right)\right] \rightarrow \frac{4x^{3/2}}{3} \text{ or } \left[\frac{4}{3}x^{3/2}\right]_1^4$$



$$= \frac{4}{3}(4)^{3/2} - \frac{4}{3}(1)^{3/2} \rightarrow \frac{32}{3} - \frac{4}{3} \rightarrow \boxed{\frac{28}{3}}$$

Ex 3:

$$\int_0^{\pi/6} \cos x\,dx \rightarrow \left[\sin x\right]_0^{\pi/6} \rightarrow \left[\overset{f(b)}{\frac{1}{2}} - \overset{f(a)}{0}\right] \rightarrow \boxed{\frac{1}{2}}$$

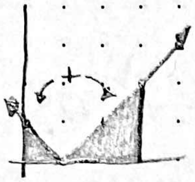


a ⟷ b "long time

$\cancel{+C}$ no see"

Ex 4: $\int_0^{\pi/4} \sec^2 x \, dx \rightarrow [\tan x]_0^{\pi/4} \rightarrow [(\overset{f(b)}{1} - \overset{f(a)}{(\frac{0}{1})}] \rightarrow \boxed{1}$

Ex 5: $\int_0^3 |x-1| \, dx$ $\quad y = a|bx-h|+k \quad \text{vertex} = (\frac{h}{b}, k)$



$\left.\begin{array}{l} \rightarrow \frac{1}{2}(1\cdot1) \rightarrow \frac{1}{2} \\ + \frac{1}{2}(2\cdot2) \rightarrow 2 \end{array}\right\} \boxed{\frac{5}{2}}$

$\int_0^1 (-x+1)\,dx + \int_1^3 (x-1)\,dx$

$\left[\frac{-x^2}{2} + x\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^3$

$\left[(-\frac{1}{2}+1) - 0\right] + \left[(\frac{9}{2}-3) - (\frac{1}{2}-1)\right] \rightarrow \frac{1}{2} + \frac{3}{2} = \boxed{\frac{5}{2}}$

Ex 6: $y = 2x^2 - 3x + 2$

$\text{Area} = \int_0^2 (2x^2 - 3x + 2)\,dx \rightarrow \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^2$

$\rightarrow \left[(\frac{2(8)}{3} - \frac{3(4)}{2} + 4) - 0\right] \rightarrow (\frac{16}{3} - \frac{12}{2} + 4) \rightarrow \frac{16}{3} - 6 + 4 \rightarrow \frac{16}{3} - \frac{6}{3} \rightarrow \boxed{\frac{10}{3}}$

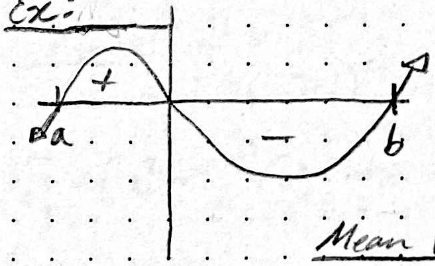
Two Special Integrals:

1. $\int_a^b f(x)\,dx = 0$ when $a = b$

2. $\int_a^b f(x)\,dx = \overset{\swarrow \text{negate}}{-c} \quad \int_b^a f(x)\,dx = c$

(when $b > a$)

Ex:



$\left.\int_a^b f(x)\,dx < 0\right\}$ when more area is under the x-axis than above it

Mean Value Theorem for Integrals

If f is continuous on $[a, b]$ and c exists between a & b such that: $\int_a^b f(x)\,dx = f(c)(b-a)$

Ex 7:

$f(x) = \sqrt{x}, \ [4, 9] \qquad \int_4^9 \sqrt{x}\,dx = f(c)(b-a)$

$\left[\frac{2x^{3/2}}{3}\right]_4^9 = \left(\frac{54}{3} - \frac{16}{3}\right) = \frac{38}{3} = f(c)(5) \rightarrow \frac{38}{3} \cdot \frac{1}{5} = f(c)$

$\frac{38}{15} = \sqrt{x}$

$\left(\frac{38}{15}\right)^2 = (\sqrt{x})^2$

$x \approx 6.418$

$\boxed{c \approx 6.418}$

Average Value of a function

$\frac{1}{b-a}\int_a^b f(x)\,dx$ ← average value is

If f is continuous on [a,b] then

Ex 8: $f(x)=2x+1$; $[-2,3]$

Average value $= \frac{1}{3+2}\int_{-2}^{3}(2x+1)\,dx = \frac{1}{5}\left[x^2+x\right]_{-2}^{3} \to \frac{1}{5}\left[(9+3)-(4-2)\right]$

$\to \frac{1}{5}[12-2] \to \frac{1}{5}(10) \to \boxed{2}$ = Average value on $[-2,3]$

4.4B Second Fundamental Theorem of Calculus:

f is continuous on I, containing "a," for every x in I:

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]=f(x)$$

$$\frac{d}{dx}\left[F(t)\right]_a^x = \frac{d}{dx}\left[F(x)-F(a)\right]$$

$$= f(x)+0 = f(x)$$

plug x into "t"

$$\frac{d}{dx}\int_{constant}^{x} f(t)\,dx = f(x)$$

e.g.

① evaluate $\frac{\cancel{d}}{\cancel{dx}}\left[\cancel{\int}_0^x \sqrt{t^2+1}\,dx\right] = \sqrt{x^2+1}$

$$\frac{d}{dx}\int_{constant}^{g(x)} f(t)\,dt = f(g(x))g'(x)$$

e.g.

$$F(x)=\int_{\pi/2}^{x^3}\cos t\,dt = \cos(x^3)(3x^2)$$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))g'(x)-f(h(x))\,h'(x)$$

4.5 Integration by Substitution

Dec, 10.2024

WU: $\int_{\pi/4}^{\pi/2} -2\csc^2 x \, dx = \left[2\cot x\right]_{\pi/4}^{\pi/2} \to \left[\left(2\cot\frac{\pi}{2}\right) - \left(2\cot\frac{\pi}{4}\right)\right] = \left(2\left(\frac{0}{1}\right)\right) - \left(2\frac{1}{1}\right) = -2$

Antidifferentiation of Composite Functions

$\int f(g(x))g'(x) \, dx = F(g(x)) + C$ letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$\int f(u) \, du = F(u) + C$

Let g be a func, range on I; f is a func., continuous on I. If g is differentiable on it's domain & F is an antiderivative of f on I, then:

Recognizing the $f(g(x))g'(x)$ pattern;

Ex 1: $\int (x^2+1)^2(2x) \, dx$

$\to u = x^2+1 \to du = 2x\,dx \to \int \underbrace{(x^2+1)^2}_{u^2}\underbrace{(2x)\,dx}_{du} \to \int u^2 du \, dx \to$

$= \frac{u^3}{3} + C \to \frac{(x^2+1)^3}{3} + C$

Ex 2: $\int 5\cos 5x \, dx$ $u = 5x$ $du = 5\,dx$

$\to \int \cos u \, du \to -\sin u + C \to \boxed{+\sin 5x + C}$

Ex 3: $\int x(x^2+1)^2 \, dx \to u = x^2+1$ $du = 2x\,dx \to$

$\frac{1}{2}\int x(x^2+1)^2(2)\,dx \to \frac{1}{2}\int u^2\,du = \frac{1}{2}\left(\frac{u^3}{2}\right) + C \to \boxed{\frac{(x^2+1)^3}{6} + C}$

Ex. 4: $\int \sqrt{2x-1} \, dx \to u = 2x-1$ $du = 2\,dx \to$

$\frac{1}{2}\int u^{1/2} du \to \frac{1}{\cancel{2}}\left(\frac{\cancel{2}u^{3/2}}{3}\right) + C \to \frac{u^{3/2}}{3} + C = \boxed{\frac{(2x-1)^{3/2}}{3} + C}$

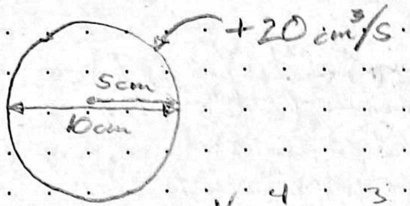
Ex 5: $\int (3-x^4)^6(4x^3)\,dx \to u = 3-x^4$ $du = -4x^3dx$

$-1\int u^6 du = -1\left(\frac{u^7}{7}\right) + C \to \frac{-u^7}{7} + C \to \boxed{\frac{-(3-x^4)^7}{7} + C}$

Related Rates Practice

Balloon inflated at +20 cm^3/s ⓐ Find $\frac{dr}{dt}$ @ 5 cm r ✓

(Sphere) ⓑ Find $\frac{dA}{dt}$ @ 5 cm r.



$\overset{\downarrow}{\frac{dV}{dt}}$

$V=\frac{4}{3}\pi r^3 \rightarrow \sqrt[3]{\frac{V}{\frac{4}{3}\pi}} = r$

$V=\frac{4}{3}\pi r^3 \rightarrow \frac{dV}{dt} = (4\pi r^2)\frac{dr}{dt} \rightarrow +20\ cm^3/s = 4\pi(\overset{\overset{r}{\downarrow}}{5})^2 \frac{dr}{dt}$

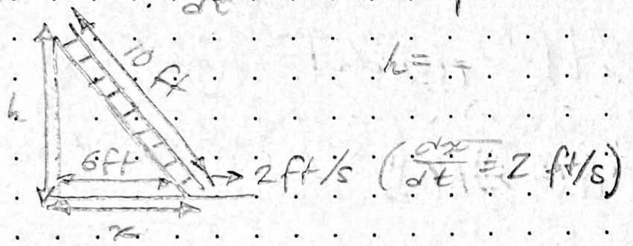
$\rightarrow 20 = 4\pi 25 \frac{dr}{dt} \rightarrow 20 = 100\pi \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{20}{100\pi} \rightarrow \frac{dr}{dt} = \boxed{\frac{1}{5\pi}\ cm/s}$ ⓐ

$A = 4\pi r^2 \rightarrow \frac{dA}{dt} = 8\pi r \frac{dr}{dt} \rightarrow \frac{dA}{dt} = 8\pi(5)(\frac{1}{5\pi}) \rightarrow \boxed{\frac{dA}{dt} = 8\ cm^2/s}$ ⓑ

Ladder (10 ft) against wall, bottom sliding away @ 2 ft/s.

ⓐ Find $\frac{dh}{dt}$ @ $x = 6 ft$.

$x^2+h^2=10^2$



h =

$\frac{d}{dt}(x^2+h^2) = \frac{d}{dt}10^2$

$\frac{2x\frac{dx}{dt} + 2h\frac{dh}{dt}}{2} = \frac{0}{2}$

$\rightarrow x\frac{dx}{dt} + h\frac{dh}{dt} = 0 \rightarrow h\frac{dh}{dt} = -x\frac{dx}{dt} \rightarrow \boxed{\frac{dh}{dt} = -\frac{x}{h}\cdot\frac{dx}{dt}}$

sub $x=6$ for h → $6^2+h^2=10^2 \rightarrow 36+h^2=100 \rightarrow h^2=64 \rightarrow h=\boxed{8}$

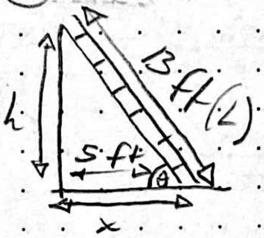
$\rightarrow \frac{dh}{dt} = -\frac{6}{8}\cdot 2\ ft/s \rightarrow \frac{dh}{dt} = \frac{-12}{8} = \boxed{-\frac{3}{2}\ ft/s}$ ⓐ

Ladder (13 ft) against wall, bottom sliding away @ 3 ft/s ($\frac{dx}{dt}$)

ⓐ Find change in ∠ between ladder & ground when $x=5 ft$

$\frac{d\theta}{dt}$

$\cos\theta = \frac{x}{L} \rightarrow \cos\theta = \frac{x}{13} \rightarrow -\sin\theta\frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$



$\rightarrow \frac{d\theta}{dt} = -\frac{1}{13\sin\theta}\frac{dx}{dt}$

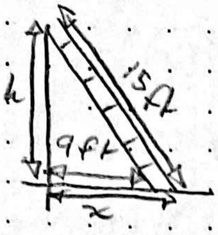
$5^2+h^2=13^2 \rightarrow 25+h^2=169$

$\rightarrow h^2=144 \rightarrow h=12\ ft$

$\rightarrow \sin\theta = \frac{12}{13} \rightarrow \frac{d\theta}{dt} = -\frac{1}{\frac{\cancel{13}}{1}\cdot\frac{12}{\cancel{13}}}(3\ ft/s) \rightarrow -\frac{1}{12}(3\ ft/s)$

$\rightarrow -\frac{3}{12} \rightarrow -\frac{1}{4} = \frac{d\theta}{dt} \leftarrow -\frac{1}{4}\ radians/s = \frac{d\theta}{dt}$

15 ft, away @ 2ft/s $\left(\frac{dx}{dt}\right)$ find $\frac{dh}{dt}$ when $x = 9$ ft



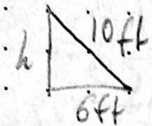
$$h^2 + 9^2 = 15^2 \rightarrow h^2 + 81 = 225 \rightarrow h^2 = 144 \rightarrow \boxed{h = 12}$$

$$h^2 + x^2 = 15^2 \rightarrow 2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0 \rightarrow 2h\frac{dh}{dt} = -2x\frac{dx}{dt}$$

$$\rightarrow h\frac{dh}{dt} = -x\frac{dx}{dt} \rightarrow \frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt} \rightarrow -\frac{9}{12}(2\text{ ft/s})$$

$$\rightarrow \frac{dh}{dt} = -\frac{18}{12} \rightarrow -\frac{9}{6} \rightarrow \boxed{-\frac{3}{2}\text{ ft/s}}$$

10 ft, away @ 1ft/s. find $\frac{dh}{dt}$ @ 6ft



$$h^2 + 6^2 = 10^2 \rightarrow h^2 + 36 = 100 \rightarrow h^2 = 64 \rightarrow h = \boxed{8}$$

$$h^2 + x^2 = 10^2 \rightarrow 2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0 \rightarrow \cancel{2}h\frac{dh}{dt} = -\cancel{2}x\frac{dx}{dt}$$

$$\frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt} \rightarrow \frac{dh}{dt} = -\frac{6}{8}\cdot(\cancel{1}\text{ft/s}) \rightarrow \frac{dh}{dt} = \boxed{-\frac{3}{4}\text{ ft/s}}$$

Fundamental theorem of Calc, II

$$\frac{d}{dx}\left[\int_c^x f(t)\,dx\right] = f(x), \qquad \frac{d}{dx}\left[\int_c^{g(x)} f(t)\,dx\right] = f(g(x))\,g'(x)$$

$$\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)\,dx\right] = (f(g(x))g'(x)) - (f(h(x))h'(x))$$

Derivaties of Trig funcs

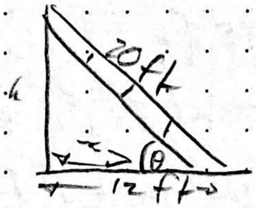
$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\sec x = \sec x\tan x \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx} = \cot x = -\csc^2 x$$

20 ft, find $\frac{d\theta}{dt}$ @ $x = 12$ ft

no $\frac{dx}{dt}$ given, ingnore



$$12^2 + h^2 = 20^2 \rightarrow 144 + h^2 = 400 \rightarrow h^2 = 256 \rightarrow \boxed{h = 16}$$

$$\cos\theta = \frac{x}{20} \rightarrow -\sin\theta\frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

$$\rightarrow \frac{d\theta}{dt} = \frac{1}{20(-\sin\theta)}\frac{dx}{dt} \rightarrow \frac{d\theta}{dt} = \frac{-1}{20\left(\frac{16}{20}\right)}\frac{dx}{dt}$$

$$\rightarrow -\frac{1}{16} \rightarrow \boxed{\frac{d\theta}{dt} = -\frac{1}{16}\text{ radians/ft}}$$

Explicit Differentiation Practice

$2y^2 - 6 = y\sin x$ $\boxed{\text{for } y>0}$

ⓐ show that $\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}$ $\rightarrow \frac{d}{dx}(2y^2 - 6) = \frac{d}{dx}(y\sin x)$

$\rightarrow 4y\frac{dy}{dx} = \left(\frac{dy}{dx}\cdot\sin x\right) + (y\cdot\cos x) \rightarrow \frac{dy}{dx}(4y - \sin x) = y\cos x$

$\rightarrow \boxed{\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}}$

ⓑ Make tangent to curve @ $(0, \sqrt{3})$

$\left.\frac{dy}{dx}\right|_{(0,\sqrt{3})} = \frac{\sqrt{3}\cos(0)}{4(\sqrt{3}) - \sin(0)} \rightarrow \frac{\sqrt{3}(1)}{4\sqrt{3} - 0} \rightarrow \frac{\sqrt{3}}{4\sqrt{3}} \rightarrow \boxed{\frac{1}{4}} = m \Big|_{(0,\sqrt{3})}$

$\boxed{y - \sqrt{3} = \frac{1}{4}(x)}$

ⓒ find $\frac{dy}{dx} = 0$ for $0 \leq x \leq \pi$

$\frac{y\cos x}{4y - \sin x} = 0 \rightarrow y\cos x = 0 \rightarrow \cos x = 0 \rightarrow \boxed{x = \frac{\pi}{2}} \rightarrow \boxed{\left(\frac{\pi}{2}, 2\right)}$

$y = 0$

$2y^2 - 6 = y\left(\sin\frac{\pi}{2}\right) \rightarrow 2y^2 - 6 = y \rightarrow 2y^2 - y - 6 = 0$

$\rightarrow y^2 - y - 12 \rightarrow \left(y - \frac{4}{2}\right)\left(y + \frac{3}{2}\right) \rightarrow (2y+3)(y-2) = 0$

$\rightarrow 2y + 3 = 0 \rightarrow 2y = -3 \rightarrow \boxed{\cancel{y = -3/2}}$ $y > 0$ $y - 2 = 0 \rightarrow \boxed{y = 2}$

ⓓ does f have rel min, max, or neither @ $\left(\frac{\pi}{2}, 2\right)$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{y\cos x}{4y - \sin x}\right) \rightarrow \frac{\left((4y - \sin x)\left(y(-\sin x) + \left(\frac{dy}{dx}\right)(\cos x)\right)\right) - \left((y\cos x)\left(4\frac{dy}{dx} - \cos x\right)\right)}{(4y - \sin x)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(\frac{\pi}{2}, 2)} \frac{\left(4(2) - \sin\frac{\pi}{2}\right)\left(2\left(-\sin\frac{\pi}{2}\right) + \left(\frac{2\cos\frac{\pi}{2}}{4(2) - \sin\frac{\pi}{2}}\right)\left(\cos\frac{\pi}{2}\right)\right) - \left(2\left(\cos\frac{\pi}{2}\right)\right)\left(4\left(\frac{2\cos\frac{\pi}{2}}{4(2) - \sin\frac{\pi}{2}}\right) - \left(\cos\frac{\pi}{2}\right)\right)}{\left(4(2) - \sin\frac{\pi}{2}\right)^2}$$

$$\rightarrow \frac{(8-1)(2(-1)) + \cancel{\left(\frac{2(0)}{8-1}\right)}(0) - \cancel{(2(0))\left(4\left(\frac{2(0)}{8-1}\right)\right)} - 0}{(8-1)^2}$$

$\rightarrow \frac{-14}{49} \rightarrow -\text{C} \rightarrow \frown$ $\boxed{\text{rel. max.}}$

@ $\left(\frac{\pi}{2}, 2\right)$